TRANSFORMATION OF SURFACES Ω

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When any surface S is referred to a conjugate system with equal point invariants, its cartesian coördinates x, y, z, are solutions of an equation of the form

$$\frac{\partial^2\theta}{\partial u\partial v}+\frac{\partial \log \sqrt{\rho}}{\partial v}\frac{\partial\theta}{\partial u}+\frac{\partial \log \sqrt{\rho}}{\partial u}\frac{\partial\theta}{\partial v}=0. \tag{1}$$

If θ_i is any solution of this equation, linearly independent of x, y and z, the surface S_i whose cartesian coördinates, x_i, y_i, z_i, are given by equations of the form

$$\frac{\partial}{\partial u}(\lambda_i x_i)=-\rho\left(x\frac{\partial\theta_i}{\partial u}-\theta_i\frac{\partial x}{\partial u}\right),\quad \frac{\partial}{\partial v}(\lambda_i x_i)=\rho\left(x\frac{\partial\theta_i}{\partial v}-\theta_i\frac{\partial x}{\partial v}\right), \tag{2}$$

where λ_i is given by the quadratures

$$\frac{\partial\lambda_i}{\partial u}=-\rho\frac{\partial\theta_i}{\partial u},\quad \frac{\partial\lambda_i}{\partial v}=\rho\frac{\partial\theta_i}{\partial v}, \tag{3}$$

is referred to a conjugate system with equal point invariants, and corresponding points M and M_i on S and S_i are harmonic with respect to the focal points of the line MM_i for the congruence of these lines. We say that S_i is obtained from S by a *transformation K*. We have studied these transformations at length in a recent memoir.[1] In the present note we consider the case where the lines MM_i form a normal congruence. In this case there exists a solution t of equation (1) such that $x^2+y^2+z^2-t^2$ also is a solution. Thus the parametric conjugate system is 2 O in the sense of Guichard, and t is the *complementary function*. The surface S_i has the same properties.

By definition a *surface C* is one possessing a conjugate system 2 O with equal point invariants. When this system is parametric, the first fundamental coefficients of C have the form

$$E=\left(\frac{\partial t}{\partial u}\right)^2+\frac{1}{\rho},\quad F=\frac{\partial t}{\partial u}\frac{\partial t}{\partial v},\quad G=\left(\frac{\partial t}{\partial v}\right)^2+\frac{1}{\rho}, \tag{4}$$

and this property is characteristic.

It can be shown that *when a surface C is referred to the system 2 O with equal point invariants, there can be found without quadratures two sur-*

faces of the same kind, say C_0 and C_0', such that the congruences of lines MM_0 and MM_0' are normal. Furthermore, the spheres of radius t with their centers on C are enveloped by two surfaces, each of which is orthogonal to one of the congruences. These orthogonal surfaces are *surfaces* Ω as defined by Demoulin,[2] who showed that they are characterized by the property that their fundamental coefficients $\bar{E}$, $\bar{G}$, $\bar{D}$, $\bar{D}''$, when the lines of curvature are parametric, satisfy the condition

$$\frac{\partial}{\partial u}\left(\frac{\frac{\partial}{\partial v}\left(\frac{\bar{D}''}{\bar{G}}\right)\sqrt{\bar{G}}}{\frac{\bar{D}}{\bar{E}}-\frac{\bar{D}''}{\bar{G}}}\,\frac{\bar{U}}{\sqrt{\bar{E}}\,\bar{V}}\right)+\frac{\partial}{\partial v}\left(\frac{\frac{\partial}{\partial u}\left(\frac{\bar{D}}{\bar{E}}\right)\sqrt{\bar{E}}}{\frac{\bar{D}}{\bar{E}}-\frac{\bar{D}''}{\bar{G}}}\,\frac{\bar{V}}{\sqrt{\bar{G}}\,\bar{U}}\right)=0, \tag{5}$$

where $\bar{U}$ and $\bar{V}$ are functions of u and v respectively.

If X, Y, Z; X_1, Y_1, Z_1; X_2, Y_2, Z_2 denote the direction-cosines of the normal to a surface C, and of the bisectors of the angles between the tangents to the parametric curves, we may write the equations of a transformation K in the form (cf. *Transactions*, loc. cit.)

$$x_i - x = \frac{1}{m_i\lambda_i}(a_iX_1 + b_iX_2 + \omega_iX), \tag{6}$$

where m_i is a constant and a_i, b_i, ω_i are functions satisfying the completely integrable system of equations

$$\begin{aligned}
\frac{\partial a_i}{\partial u} &= -m_i(\lambda_i - \rho\theta_i)\sqrt{E}\cos\omega + b_iA + \frac{w_iD}{2\sqrt{E}\cos\omega},\\
\frac{\partial a_i}{\partial v} &= -m_i(\lambda_i + \rho\theta_i)\sqrt{G}\cos\omega - b_iB + \frac{w_iD''}{2\sqrt{G}\cos\omega},\\
\frac{\partial b_i}{\partial u} &= \ \ m_i(\lambda_i - \rho\theta_i)\sqrt{E}\sin\omega - a_iA = \frac{w_iD}{2\sqrt{E}\sin\omega},\\
\frac{\partial b_i}{\partial v} &= -m_i(\lambda_i + \rho\theta_i)\sqrt{G}\sin\omega + a_iB + \frac{w_iD''}{2\sqrt{E}\sin\omega},\\
\frac{\partial w_i}{du} &= -\frac{D}{2\sqrt{E}}\left(\frac{a_i}{\cos\omega} - \frac{b_i}{\sin\omega}\right),\quad \frac{\partial w_i}{dv} = -\frac{D''}{2\sqrt{G}}\left(\frac{a_i}{\cos\omega} + \frac{b_i}{\sin\omega}\right),
\end{aligned} \tag{7}$$

where 2ω is the angle between the parametric lines on C, D and D'' are the second fundamental coefficients of C, and

$$A = \sqrt{\frac{E}{G}}\,\frac{\partial \log\sqrt{\rho}}{\partial v}\sin 2\omega - \frac{\partial\omega}{\partial u},\quad B = \sqrt{\frac{G}{E}}\,\frac{\partial \log\sqrt{\rho}}{\partial u}\sin 2\omega - \frac{\partial\omega}{\partial v}.$$

For the sake of brevity we put

$$T_i^2 = a_i^2 + b_i^2 + w_i^2, \quad H = \sqrt{EG - F^2}. \tag{8}$$

The functions $a_0, b_0, w_0, \lambda_0, \theta_0$ which determine C_0 are given by

$$a_0 = - \sin\omega \cdot w_0 \rho \left(\sqrt{G}\frac{\partial t}{\partial u} + \sqrt{E}\frac{\partial t}{\partial v}\right),$$

$$b_0 = \cos\omega \cdot w_0 \rho \left(\sqrt{G}\frac{\partial t}{\partial u} - \sqrt{E}\frac{\partial t}{\partial v}\right),$$

$$\frac{\partial \log w_0}{\partial u} = \frac{D}{H}\frac{\partial t}{\partial u}, \quad \frac{\partial \log w_0}{\partial v} = \frac{D''}{H}\frac{\partial t}{\partial v}, \quad T_0 = H\rho w_0, \tag{9}$$

$$\frac{m_0}{T_0}(\lambda_0 - \rho\theta_0) = \frac{D}{H} + \frac{1}{\frac{\partial t}{\partial u}}\frac{\partial}{\partial u}\log H\rho, \quad \frac{m_0}{T_0}(\lambda_0 + \rho\theta_0) = \frac{D''}{H} + \frac{1}{\frac{\partial t}{\partial v}}\frac{\partial}{\partial v}\log H\rho.$$

The complementary function t_0 for C_0 is given by

$$t_0 = t - \frac{T_0}{\lambda_0 m_0}. \tag{10}$$

The functions for the surface C_0' are analogous to the above.

Ordinarily the surfaces S_i derived from a surface C by transformations K are not surfaces C. However, the equations

$$\begin{aligned} &\frac{\partial\theta_1}{\partial u} + \sqrt{E}\left[\cos\omega a_1 - \sin\omega b_1 + (t_1 - t)\frac{m_1\lambda_1}{T_0}(\cos\omega a_0 - \sin\omega b_0)\right] \\ &\frac{\partial\theta_1}{\partial v} + \sqrt{G}\left[\cos\omega a_1 + \sin\omega b_1 + (t_1 - t)\frac{m_1\lambda_1}{T_1}(\cos\omega a_0 + \sin\omega b_0)\right] \end{aligned} \tag{11}$$

are consistent with equations (7) for $i = 1$, the function θ_1 so defined is a solution of equation (1), and the new surface S_1, given by (6), is a surface C, say C_1. In particular we remark that the function t_1 given by (2), when x_i and x are replaced by t_1 and t respectively, is the complementary function for C_1. Furthermore, if x_i and x in equations (2) are replaced by $x_1^2 + y_1^2 + z_1^2 - t_1^2$ and $x^2 + y^2 + z^2 - t_1^2$ the resulting equations are satisfied.

With the aid of the theorem of permutability of general transformations K (cf. *Transactions*, loc. cit., p. 406) we show that *if C, C_0 and C_1 are three surfaces in the relation indicated above, a fourth surface C_{10} can be found without quadratures such that the lines joining corresponding points M_1, M_{10}, on C_1 and C_{10} is a normal congruence.*

Likewise it is found that *the surfaces Ω and Ω_1 normal to the congruences of the lines MM_0 and $M_1 M_{10}$ at the distances t and t_1 from C and C_1 respectively envelope a two-parameter family of spheres, and the lines of* *curvature on* Ω *and* Ω_1 *correspond.* Thus equations (7) and (11) define transformations of surfaces Ω of the Ribaucour type. We call them *transformations* A. When in particular the surfaces C and C_1 are *associate* surfaces, which is a special case of transformations K, the surfaces C_0 and C_{10} are likewise associate, and the surfaces Ω and Ω_1, as defined in the preceding theorem, have the same spherical representation of their lines of curvature.

By means of a generalized theorem of permutability for transformations K in general we prove the following theorem of permutability for transformations A: *If* Ω_1 *and* Ω_2 *are two surfaces obtained from a surface* Ω *by transformations* A, *there exists a surface* Ω' *which is in the relations of transformations* A *with* Ω_1 *and* Ω_2, *and* Ω' *can be found without quadratures.*

Isothermic surfaces are surfaces C for which $t = 0$. In this case the transformations A are equivalent to the transformations D_m of isothermic surfaces, discovered by Darboux and studied at length by Bianchi.[3]

Surfaces with isothermal representation of their lines of curvature are surfaces Ω in the sense that the surface C is the locus of the point midway between the centers of principal curvature of Ω, and C_0 is at infinity. This case requires special study, but the results are analogous to those of the general case. However, the transformations A are now the same as the transformations of these surfaces established from another point of view by me.[4]

[1] Eisenhart, *Trans. Amer. Math. Soc.* **15**, 397–430 (1914).
[2] Demoulin, *Paris, C. R. Acad. Sci.*, **153**, 703 (1911).
[3] Bianchi, *Annali Mat. Milano*, Ser. 3, **11**, 93–158 (1905).
[4] Eisenhart, *Trans. Amer. Math. Soc.*, **9**, 149–177 (1908).

POTASSIUM AMMONO ARGENATE, BARATE, CALCIATE, AND SODATE

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It has been shown by me that reactions strictly analogous to those which accompany the solution of the hydroxides of zinc, lead, and aluminium in aqueous solutions of potassium hydroxide take place when the amides of certain metals are treated with liquid-ammonia solutions of potassium amide. Thus, just as zinc hydroxide is known to dissolve in an aqueous solution of potassium hydroxide to form potassium aquo-